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On the Zero-Energy-Limit Solution for the Modified Gross–Pitaevskii Equation

Kwang-Hua W. Chu

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Abstract The modified Gross–Pitaevskii equation was derived and solved to obtain the 1D solution in the zero-energy limit. This stationary solution could account for the dominated contributions due to the kinetic effect as well as the chemical potential in inhomogeneous Bose gases.

Keywords Zero-point energy · Quantum fluctuations

1 Introduction

Studies of collision phenomena in rather cold gases, e.g., dilute Bose gases, have recently attracted many researchers' attention [10]. One relevant research interest is about the solution of the appropriate and modified Gross–Pitaevskii equations for different dimensions [4, 10]. New possibilities for observation of macroscopic quantum phenomena arises because of the recent realization of Bose–Einstein condensation in atomic gases [10]. There are two important features of the system-weak interaction and significant spatial inhomogeneity. Because of this inhomogeneity a non-trivial *zeroth-order* theory exists, compared to the *first-order* Bogoliubov theory. This theory is based on the mean-field Gross–Pitaevskii equation for the condensate ψ-function. The equation is classical in its essence but contains the $(h/2p)$ constant explicitly. Phenomena such as collective modes, interference, tunneling, Josephson-like current and quantized vortex lines can be described using this equation. The study of deviations from the zeroth-order theory arising from zero-point and thermal fluctuations is also of great interest [7, 9, 11]. Thermal fluctuations are described by elementary excitations which define the thermodynamic behavior of the system and result in Landau-type damping of collective modes.

As a preliminary attempt, following the mean-field approximate formulation in [10], in this short paper, we plan to investigate the 1D solution for the modified Gross–Pitaevskii

K.-H.W. Chu (✉)
College of Physics and Information Engineering, Hebei Normal University, Shijiazhuang 050016, China
e-mail: khwchu@126.com

equation in the zero-energy limit. This presentation will give more clues to the studies of the quantum non-equilibrium thermodynamics in inhomogeneous (dilute) Bose gases and the possible appearance of the kinetic mechanism before and/or after Bose–Einstein condensation which is directly linked to the particles (number) density and their energy states or chemical potentials.

2 Formulations

The generalization of the Bogoliubov prescription [2] for the ψ-operator to the case of a spatially nonuniform system is

$$\hat{\psi}(\mathbf{r},t) \approx \psi_0(\mathbf{r},t) + \hat{\phi}(\mathbf{r},t) \tag{1}$$

where ψ_0 is the condensate wave function. This is an expression of the second quantization (ψ-operator) for atoms as $n_0 = N_0/V = |\psi_0|^2$ and $\phi \ll \sqrt{n_0}$, N_0 is the number of atoms in the condensate. To neglect $\hat{\phi}$ means neglecting all correlations and this is a poor approximation when distances between particles are of the order of the effective radius r_e of the atom-atom interaction. To overcome this problem the assumption that the atomic gas is dilute: $nr_e^3 \ll 1$ should be made [10] ($n = N/V$, N is the number of atoms confined in V). We can use the procedure of the quantum virial expansion to calculate the energy of the system. We have the form

$$E = E_0 + \frac{g}{2}\int n^2(\mathbf{r})d\mathbf{r}, \tag{2}$$

for the energy of slow particles, where E_0 is the energy of the gas without interaction, $n(\mathbf{r})$ of the density of the gas, $g = 4\pi a\hbar^2/m$, a is the s-wave scattering length and m is the particle's mass [10]. After taking into account the correlations (in above equation so that we can neglect $\hat{\phi}$) and considering E as an *effective Hamiltonian* we then have the celebrated Gross–Pitaevskii equation

$$i\hbar\frac{\partial}{\partial t}\psi_0(\mathbf{r},t) = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + g|\psi_0(\mathbf{r},t)|^2\right]\psi_0(\mathbf{r},t) \tag{3}$$

which describe the dynamics of a non-uniform nonideal Bose gas at $T = 0$. Here, $V_{\text{ext}}(\mathbf{r})$ is the confining potential. If the gas is in its ground state, the time dependence of ψ_0 is given by $\psi_0 \sim \exp(-i\mu t/\hbar)$, where μ is the chemical potential of the gas [10]. We thus obtain the form

$$\left[-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + g|\psi_0(\mathbf{r},t)|^2 - \mu\right]\psi_0(\mathbf{r},t) = 0 \tag{4}$$

which could be stationary once t is fixed or selected. We noticed that an equation of above form has been considered before in connection with the theory of superfluidity of liquid helium close to the λ-point [6].

To investigate our interest here, we shall consider the 1D solution of (4) for the case of zero-energy limit. Firstly we consider a 3D Bose gas confined tightly in one dimension and weakly in the remaining two dimensions on a length scale l_t ($= \sqrt{\hbar/2m\omega}$ for a harmonic trap of angular frequency of ω). A collision between two condensate particles will typically occur over the characteristic length scale l_{col} once l_t is much larger than l_{col} (thus we can use a local density approximation).

We now model the pair wavefunction of two atoms in the medium by that a single particle with the reduced mass moving in a potential which consists of a circularly symmetric box of radius R and a hard sphere of radius R_a located in the centre of the box. Following the reasoning in the derivation of (1), we introduce a similar bias or ghost-effect for $\hat{\phi}$: Ψ which can be relevant to certain critical or kinetic (non-equilibrium) effect (or configurational dissipation, quantum fluctuations included) [7, 8, 11, 12] so that $\mu\Psi = \varepsilon$ in the zero-energy, zero-momentum limit. It is presumed that ε still reaches zero in the homogeneous limit. The problem for a 2D $\psi(r,\theta)$ becomes, after referencing to the bias Ψ,

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{\partial \theta^2}\right]\psi = -\varepsilon \tag{5}$$

and, in fact, for a circularly symmetric $\psi(r)$,

$$\left[\frac{d^2}{dr^2} + \frac{d}{rdr}\right]\psi = -\varepsilon \tag{6}$$

with the boundary conditions: the wave function vanishes on the inner radius ($\psi = 0$ as $r = R_a$), and reaches an asymptotic value at the edge of the box ($\psi \to \Pi$ as $r = R$).

We can obtain the solution

$$\psi(r) = \frac{\varepsilon}{4}\left[R^2 - r^2 + (R^2 - R_a^2)\frac{\ln(r/R)}{\ln(R/R_a)}\right] + \Pi\frac{\ln(r/R_a)}{\ln(R/R_a)}, \tag{7}$$

where $R_a \le r \le R$. The extra energy caused by the curvature of this wave function resulting from the scattering potential is

$$\begin{aligned}\Delta E &= \frac{\hbar^2}{2m}\int_0^{2\pi}\int_{R_a}^{R} |\nabla\psi(r)|^2 r\,dr\,d\theta \\ &= \pi\frac{\hbar^2}{m}\left\{\varepsilon\left[\frac{R^3 - R_a^3}{12} + \varepsilon\frac{R^2 - R_a^2}{\ln(R/R_a)}\left(\frac{1}{16} - \frac{R - R_a}{4}\right)\right]\right. \\ &\quad \left. + \frac{\Pi}{\ln(R/R_a)}\left[\Pi + \varepsilon(R - R_a)\left(\frac{R - R_a}{2} - 1\right)\right]\right\}.\end{aligned} \tag{8}$$

3 Results and Discussions

Note that, this (extra) energy depends upon the size of the box R, which is indeed the length scale relevant for the scattering of two particles in two dimensions [10]. The scattering of two particles in a many-body system should obviously not depend on the size of the system as a whole when R becomes large, and so we must interpret R as the physical relevant length scale l_{col}. The appropriate length scale over which a many-body wavefunction changes is the healing length l_h, given in homogeneous Bose condensed systems by $l_h = \hbar/\sqrt{2mg_{2D}n_0} = \hbar/\sqrt{2m\mu}$ [10], and so it is this which must be used in (8). We recall that $|\Pi|^2$ corresponds to the condensate density n_0 and the homogeneous limit for a pair interaction strength can still be recovered from above equation.

From (7), We can obtain and observe the relationship for $\psi(r)$ w.r.t. r (in terms of units of l_t) for different ε (say, 0.0005, 0.001, 0.005, 0.01) under the same Π (say, 0.01). The resulting presentation shows the significant effect of the kinetic part (say, Π) due to ε.

The effect of boundary conditions, like Π is minor. From the definition of $\varepsilon = \mu\Psi$, we can understand that the contribution of the chemical potential for ψ of the inhomogeneous gases is indeed dominated, too. We shall investigate more complicated problems [1, 3, 5] in the future.

References

1. Adhikari, S.K., Muruganandam, P.: J. Phys. B At. Mol. Opt. Phys. **35**, 2831 (2002)
2. Bogoliubov, N.N.: J. Phys. USSR **11**, 23 (1947)
3. Chu, K.-H.: Post-Dr. Report. Peking University, Beijing (2001) (in English)
4. Chu, K.-H.W.: Preprint (2002)
5. Chu, A.K.-H.: Int. J. Mod. Phys. B **21**, 1 (2007)
6. Ginzburg, V.L., Pitaevskii, L.P.: Sov. Phys. JETP **7**, 858 (1958)
7. Huang, K.S.: Phys. Rev. Lett. **83**, 3770 (1999)
8. Kagan, Yu.M., Svistunov, Shlyapnikov, G.V.: Sov. Phys. JETP **74**, 279 (1992)
9. Penrose, O., Onsager, L.: Phys. Rev. **104**, 576 (1956)
10. Pitaevskii, L.P.: Int. J. Mod. Phys. B **13**, 427 (1999)
11. Salasnich, L.: Int. J. Mod. Phys. B **15**, 1253 (2001)
12. Snoke, D.W., Wolfe, J.P.: Phys. Rev. B **39**, 4030 (1989)